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Note

Chebyshev Expansion Methods for the Solution of Elliptic Partial Differential Equations

In a recent paper, Haidvogel and Zang (*J. Comput. Phys.* **30** (1979), 167) described a method, based on an expansion in Chebyshev polynomials, for the solution of Poisson's equation in a rectangle. They also indicated how the method might be extended to more general equations. In this note, we give a comparison of the method with the global element method (Delves and Hall, *J. Inst. Math. Appl.* **23** (1979), 223), and give numerical results for those problems treated by Haidvogel and Zang.

1. The Method of Haidvogel and Zang

In a recent paper [1], hereafter referred to as HZ, Haidvogel and Zang advocated the use of methods based on expansions in Chebyshev polynomials, for the accurate solution of elliptic partial differential equations, and gave a detailed description of one such algorithm for the solution of Poisson's equation in a square:

$$\nabla^2 U(x, y) = f(x, y), \qquad -1 \leqslant x, y \leqslant 1, \tag{1}$$

subject to homogeneous boundary conditions

$$U(x, y) = 0, \qquad |x| = 1 \text{ or } |y| = 1. \tag{2}$$

The method they described involves the expansion of both U and f as truncated double Chebyshev series:

$$\begin{aligned} U(x, y) &\approx U_N(x, y) = \sum_{n=0}^{N} \sum_{m=0}^{N} a_{nm} T_n(x) T_m(y), \\ f(x, y) &\approx \sum_{n=0}^{N} \sum_{m=0}^{N} f_{nm} T_n(x) T_m(y). \end{aligned} \tag{3}$$

Approximate values for the coefficients f_{nm} can be computed numerically using fast Fourier transform techniques. Applying the Laplace operator to the expansion for U, and equating coefficients of $T_n T_m$ in (1) and (2), yields an $(N+1) \times (N+1)$ square set of algebraic equations for a_{nm} in terms of the f_{nm}, and HZ describe efficient methods of solution of these defining equations. They also comment that the method extends readily to handle both inhomogeneous boundary conditions, and the Helmholtz equation

$$\nabla^2 U + \Omega U = f(x, y), \qquad \Omega \text{ constant}, \tag{4}$$

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and indicate further how the method might be extended to handle problems of the form

$$\nabla \cdot a(x, y) \nabla U = f. \tag{5}$$

The chief interest in such methods lies in the very rapid convergence which can be obtained for smooth problems. Provided that the exact solution $U(x, y)$ is infinitely differentiable on the closed domain $-1 \leqslant x,\ y \leqslant 1$, exponentially fast convergence can be expected; that is, the error norm $\|U_N - U\|$ will reduce at the rate

$$\|U_N - U\| \approx Ca^N, \qquad C \text{ a positive constant, } 0 < a < 1. \tag{6}$$

This compares with an expected behaviour for a finite difference method of order p and a rectangular mesh of $N \times N$ points of

$$\|U_N - U\| \approx C'N^{-p}. \tag{7}$$

For large N, (6) represents very much more rapid convergence than (7) for any finite p. That this rapid convergence is indeed attained in practice is shown by the results of example 1 of HZ; see also below. Unfortunately, singularities of various types occur very commonly in the solution of elliptic partial differential equations; when they occur, the exponential convergence form (6) is lost and the convergence rate achieved by the HZ algorithm reduces to the power form (7), the value of p depending on the severity of the singularity. This is demonstrated by the second and third examples given in HZ. The second example has a solution with a "mildly non-analytic" behaviour near the corners of the square, leading to a convergence rate of the form (7) with $p \approx 6$; the third example involves an interface problem for which $f(x, y)$ is discontinuous, and leads to convergence of the form (7) with $p \approx 2$; for this last example, the method of HZ converges no faster than a standard five-point finite difference method. These examples are considered further below.

The failure to provide a mechanism for treating singularities drastically limits the applicability of a straightforward Chebyshev method such as that of HZ. Further, HZ give no mechanism for treating other than rectangular domains (although such global expansion methods can be extended: see [6, 7]).

2. The Global Element Method

The global element method [2] is also an expansion method based primarily on orthogonal polynomials. It is, however, rather more general than that of HZ, being designed to handle the variable coefficient second order self-adjoint equation

$$-\frac{\partial}{\partial x}\left(A_{11}(x, y)\frac{\partial U}{\partial x}\right) - \frac{\partial}{\partial x}\left(A_{12}(x, y)\frac{\partial U}{\partial y}\right) - \frac{\partial}{\partial y}\left(A_{12}(x, y)\frac{\partial U}{\partial x}\right)$$
$$-\frac{\partial}{\partial y}\left(A_{22}(x, y)\frac{\partial U}{\partial y}\right) + B(x, y)\, U(x, y) = f(x, y) \tag{8}$$

with a variety of homogeneous or inhomogeneous boundary conditions, on a general curvilinear region R. An extension to non-self-adjoint problems is given in [8]. Briefly, the method proceeds as follows:

(i) The problem region is divided into a small number M of sub-regions ("global elements"); in the rth element an expansion of the form

$$U(x, y) \approx U_{pr}(x, y) = \sum_{i=1}^{p} a_i^{(r)} h_i^{(r)}(x, y) \tag{9}$$

is taken for the solution $U(x, y)$.

(ii) Problem (8) is viewed as a set of M coupled problems; coupling between elements being via continuity conditions which must be satisfied across element boundaries.

(iii) The coefficients $a_i^{(r)}$ are determined as the stationary point of a variational functional; see [2] for details. The functional used treats both differential equation and boundary conditions on an equal footing, and does not impose any subsidiary conditions (other than regularity conditions) on the approximate solution. Hence both differential equation and all boundary conditions, including Dirichlet conditions, are automatically satisfied approximately. Further, since the interface conditions are treated as boundary conditions, these too are satisfied approximately; thus, in particular, the approximate solution is *not* continuous across element boundaries (i.e., it is "nonconforming"), although it is "approximately continuous."

The defining equations for the coefficients $a_i^{(r)}$ involve integrals over the subregions used; in two dimensions, with a polynomial basis of degree $N-1$ and M elements, an $MN^2 \times MN^2$ matrix results which is block-sparse (blocks corresponding to non-adjacent elements are empty). Non-empty blocks are in general full; and the diagonal blocks involve the accumulation of MN^2 double integrals. The economics of the scheme do not therefore appear promising at first sight. However, for the range of problems covered by HZ (rectangular region; Laplace operator) and with a polynomial basis, all of the integrals can be performed analytically. More generally, an efficient implementation, which avoids the direct numerical evaluation of integrals and takes account of the structure of the equations during their solution, was given in [3]. This implementation has the following features:

(iv) Each element is mapped onto the square $[-1, 1] \times [-1, 1]$ using a "blending function" map [5]. In the mapped coordinates (s, t), the solution within an element is taken to have the form of a modified Chebyshev expansion,

$$U_N(s, t) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} h_i(s) h_j(t), \tag{10}$$

where

$$h_1(s) = 1; \qquad h_2(s) = s; \qquad h_i(s) = (1 - s^2)\, T_{i-2}(s), i \geqslant 2.$$

The particular form of (10) is chosen for computational efficiency (see [3]), and the user sees only a standard Chebyshev expansion:

$$U_N(s, t) = \sum_{i=0}^{N-1}{}'' \bar{a}_{ij} T_i(s)\, T_j(t). \tag{11}$$

It is shown in [3] that the choice (10) allows the defining equations to be set up directly from Chebyshev expansions of the coefficients $A_{ij}(x, y)$, $B(x, y)$, $f(x, y)$ in (8); these expansions are computed numerically in the mapped coordinates (s, t), using FFT techniques. This indirect approach reduces the cost of setting up the defining equations to $\mathcal{O}(MN^4)$; an iterative scheme for solving these equations is also given in [3], with an operation cost which is also $\mathcal{O}(MN^4)$.

The global element method has a number of similarities with the method of HZ. However, it also has a number of differences, of which the two most important are its treatment of the region, and of singularities:

(a) *Region*

The method of HZ produces the solution as a single Chebyshev expansion over the whole problem domain. Even when the solution is analytic over this region, it is simple to display problems for which a large number of terms are required to provide an initial representation of the solution, even though eventually convergence is very rapid [a simple one-dimensional example of such a problem is one with solution $e^{\alpha x}$ on the domain $[-1, 1]$; for large α a very high degree polynomial is required to represent this function, even though eventually convergence is exponentially fast].

In such cases, it can be more efficient to split the region into two or more subregions, and use a separate expansion on each; the global element method allows naturally for this. The blending function maps used do not introduce extraneous singularities into the equations (provided that the subregions have no cusps, that is, corners with zero corner angle), and have the advantages of being explicit, and of treating curved sides exactly. Subregions with three and four sides are covered by the implementation described in [3].

(b) *Treatment of Singularities*

Two types of singularities are common in practice:

(i) Line singularities: one or more of the coefficients in the differential equation is discontinuous across a given line in the region R. Such discontinuities typically arise, for example, at the interface between two regions with different physical properties. They are treated in the global element method merely by making the line of discontinuity a subregion boundary; see example 3 below.

(ii) Point singularities: the solution exhibits some non-analytic behaviour at a point or points in the domain. These are treated by isolating the point of singularity within a subregion, and constructing a singular map for this subregion chosen so that the solution is *smooth* when expressed in terms of the mapped variable. For an example of such a calculation, see [4].

3. Comparison of the Methods

Although the approximating equations for the two methods are constructed differently, those for the global element being constructed directly from a variation principle, the method of HZ should yield similar accuracy for a given value of N to that given by a single-element global element calculation. However, in addition to its ability to handle curved boundaries and the general equation (8), the ability to subdivide the region, and to give a direct treatment of interface problems and of point singularities, can be expected to be of advantage in many cases. We illustrate this by presenting the global element solution of the three examples used by Haidvogel and Zang to illustrate their method. The region for each of these examples is the square $R: -1 \leqslant x,\ y \leqslant 1$, and the boundary conditions are homogeneous ($U(x, y) = 0$ on the boundary). We present results obtained by a subdivision of this region into four unit squares by the x and y axes, together with a linear map of each subregion onto the "standard" square $[-1, 1] \times [-1, 1]$. Within each element a polynomial solution of degree $N - 1$ is constructed, giving a total of $4N^2$ unknown coefficients; we therefore compare the accuracy achieved with that obtained by HZ with a polynomial solution of degree $2N$, giving a total of $(2N + 1)^2$ unknown coefficients, and for the case in which their equation solver is iterated to convergence.

For the global element method we report both the maximum observed error, and the error estimate returned by the program. The error estimate is based on an analysis of the convergence of the computed Chebyshev expansion, and a similar estimate could be provided within the method of HZ.

The program used, GEM 2, is still at an early stage of development, and has a number of limitations. The most serious is that it does not yet implement the iterative solution scheme of [3], but uses instead a block Gauss elimination scheme. The resulting storage requirements limit the maximum degree approximations which can be handled to those shown in the examples below; the paper by HZ gives extended results for larger values of N, and these are shown here for completeness. The program GEM2 is also still rather slow, and we do not attempt here to compare solution times with those given by HZ; these times would in any case have little relevance, since the technique of HZ is optimised for the Laplace operator and square region of these examples, and a similarly optimised global element procedure should therefore be provided if meaningful speed comparisons are to be made. Rather, we are concerned here with demonstrating that the facility to subdivide the given region can yield substantially more rapid convergence than the global approach of HZ.

PROBLEM 1.

$$\nabla^2 U = -32\pi^2 \sin 4\pi x \sin 4\pi y. \tag{12}$$

TABLE I

Computed Results for Problem 1, for the Global Element Method Using Four Elements and an Approximation of Degree $N-1$, and for the Method of HZ [1] Using an Approximation of Degree $2N$.

	Global element		HZ
N	Max. error	Est. error	
6	4.8×10^{-1}	2.3×10^{-1}	
7	1.5, −1	8.7, −2	
8	1.1, −2	7.7, −3	3.3, −2
9	7.1, −3	3.9, −3	
10	9.3, −5	8.6, −5	
12	4.5, −7	4.9, −7	6.9, −6
16			4.8, −11
24			1.9, −12
32			8.7, −13

TABLE II

	Global element		HZ
N	Max. error	Est. error	Max. error
3	8.6×10^{-3}	1.8×10^{-1}	
4	1.6, −3	9.8, −3	
5	3.5, −4	1.7, −3	
6	7.8, −5	2.1, −4	
7	7.4, −6	6.4, −5	
8	7.2, −6	2.3, −5	3.5×10^{-5}
9	3.4, −6	1.0, −5	
10	8.9, −7	4.8, −6	
12	2.9, −7	1.3, −6	
16			2.2, −6
32			1.4, −7
64			8.7, −9

Exact solution:

$$U(x, y) = \sin 4\pi x \sin 4\pi y.$$

The inhomogeneous term in this equation is analytic in R, and so is the solution. Both methods are therefore expected to converge exponentially fast. There might, however, still be an advantage in subdividing the region, to pick up the structure (oscillations) in the right hand side of (12). The results obtained are shown in Table I. Both methods converge very rapidly, as expected; there appears to be a minor, unimportant advantage in subdividing the region, for this problem.

PROBLEM 2. $\nabla^2 U = 1$.

Exact solution: $U(x, y) = U(y, x)$

$$= -\frac{1}{2}(1-y^2) + \frac{16}{\pi^3} \sum_{\substack{m=1 \\ m \text{ odd}}} \left[\frac{(-1)^{(m-1)/2}}{m^3(1+e^{-m\pi})} \right.$$
$$\left. \times \cos \frac{1}{2} m\pi \, y e^{-m\pi(1-|x|)/2} (1 + e^{-m\pi|x|}) \right]. \qquad (13)$$

The coefficients in this equation are again analytic, but the solution exhibits a mild non-analyticity near the corner of the square, approaching zero as $r^2 \ln r$, where r is the distance from a corner. The convergence achieved by the method of HZ is therefore algebraic rather than exponential. The global element solution presented here makes no attempt to map away this singularity, and hence is expected also to converge only algebraically. The results obtained are given in Table II. Those for the GEM show an odd–even effect in N, reflecting the even symmetry of the solution about the lines $x = 0$, $y = 0$. Allowing for this feature, a fit to the results indicates that the convergence is quite well modelled by (7), with parameters

$$p = 4 \qquad \text{(HZ)},$$
$$p = 8 \qquad \text{(GEM)}.$$

This higher rate of convergence obtained using the GEM shows very clearly in the results, but is a little surprising. It may possibly stem from the difference between the variational formalism used in the GEM, and the more straightforward truncation of the infinite equations used in HZ; or it may only imply that the values of N used are not large enough for the asymptotic formula (7) to be valid. However, it is clear from the rapid convergence achieved that solutions of a given accuracy can be obtained using fewer unknowns with the GEM formalism with a subdivision of the region, than by the whole region expansions of HZ.

TABLE III

	Global element		HZ
N	Max. error	Est. error	
3	2.5×10^{-8}	4.7×10^{-2}	
4	5.1, −10	1.6, −10	
5	2.0, −10	2.6, −11	
8			5.2×10^{-4}
16			1.3, −4
32			3.0, −5
64			7.4, −6

PROBLEM 3.

$$\nabla^2 U = H(x)\, v(y) + H(y)\, v(x). \tag{14}$$

$$\begin{aligned} H(x) &= 0, && x < 0, \\ &= \tfrac{1}{2}, && x = 0, \\ &= 1, && x > 1, \\ v(x) &= -\tfrac{1}{4}(x+1), && x < 0, \\ &= \tfrac{1}{2}x^2 - \tfrac{1}{4}x - \tfrac{1}{4}, && x \geqslant 0. \end{aligned}$$

Exact solution: $U(x, y) = v(x)\, v(y)$.

This example gives a simple model of an interface problem, the interfaces lying along the lines $x = 0$, $y = 0$. The exact solution has discontinuous second derivatives across these lines, and this results in slow convergence for the method of HZ. The GEM, however, finds no difficulty, since the interfaces lie on element boundaries. As shown in Table III, the GEM obtains essentially machine accuracy (10–11 decimal digits) with $N \geqslant 4$; and this small error is reflected in the error estimates returned by the programme. For this simple problem, the finite difference methods tested for comparison by HZ would also obtain essentially machine accuracy, provided that the interfaces were made grid lines for the mesh used. This was explicitly avoided by HZ, since for more realistic problems, with curved interfaces, it is difficult to achieve within a finite difference framework. However, the GEM has no such difficulty, and indeed the accurate treatment of interface problems was one of its design aims.

4. Conclusions

The results presented strengthen the conclusion of HZ that Chebyshev expansion methods are attractive for the accurate solution of elliptic problems. They also suggest that there is considerable advantage to be gained, in flexibility and accuracy, by allowing for a subdivision of the region.

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